Introducing Proof Tree Automata and Proof Tree Graphs

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LORIA, Université de Lorraine

Eleventh Scandinavian Logic Symposium
17 June 2022

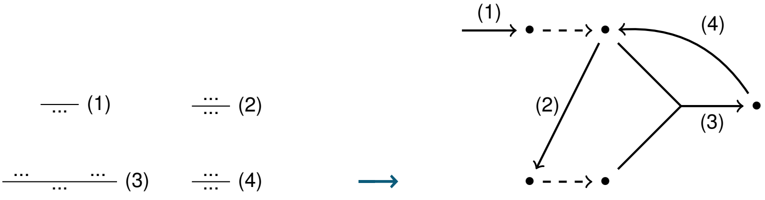
What this talk is about

$$\frac{}{\ldots}\ (1) \qquad \frac{\ldots}{\ldots}\ (2)$$

$$\frac{\ldots \qquad \ldots}{\ldots}\ (3) \qquad \frac{\ldots}{\ldots}\ (4)$$

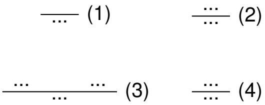
Calculus $\mathcal{K}$

What this talk is about

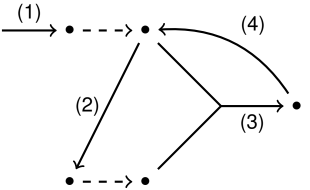


Calculus $\mathcal{K}$ Graph $\mathcal{G}$

What this talk is about



Calculus $\mathcal{K}$

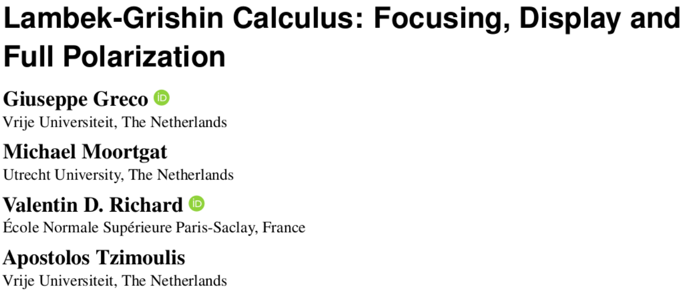


Graph $\mathcal{G}$

Graphical representation of automaton $\mathcal{A}$

Working on a huge calculus

G. Greco



A lot of rules!

$$\frac{}{p \vdash^{+} p}\; p\text{-Id} \qquad n\text{-Id}\; \frac{}{n \vdash^{-} n}$$

$$\text{P-Cut}\; \frac{\mathring{X} \mathring{\vdash}^{+} \mathring{P} \qquad \mathring{P} \mathring{\vdash}^{+} \mathring{Y}}{\mathring{X} \mathring{\vdash}^{+} \mathring{Y}} \qquad \frac{\mathring{\Gamma} \mathring{\vdash}^{-} \mathring{N} \qquad \mathring{N} \mathring{\vdash}^{-} \mathring{\Delta}}{\mathring{\Gamma} \mathring{\vdash}^{-} \mathring{\Delta}}\; \text{N-Cut}$$

$$\text{Pn-Cut}\; \frac{\mathring{X} \mathring{\vdash}^{+} \mathring{P} \qquad \mathring{P} \mathring{\vdash} \mathring{\Delta}}{\mathring{X} \mathring{\vdash} \mathring{\Delta}} \qquad \frac{\mathring{X} \mathring{\vdash} \mathring{N} \qquad \mathring{N} \mathring{\vdash}^{-} \mathring{\Delta}}{\mathring{X} \mathring{\vdash} \mathring{\Delta}}\; \text{nN-Cut}$$

$$\otimes_L\; \frac{\mathring{P} \hat{\otimes} \mathring{Q} \mathring{\vdash} \mathring{\Delta}}{\mathring{P} \otimes \mathring{Q} \mathring{\vdash} \mathring{\Delta}} \quad \frac{\mathring{X} \mathring{\vdash}^{+} \mathring{P} \qquad \mathring{Y} \mathring{\vdash}^{+} \mathring{Q}}{\mathring{X} \hat{\otimes} \mathring{Y} \vdash^{+} \mathring{P} \otimes \mathring{Q}}\; \otimes_R \qquad \oplus_L\; \frac{\mathring{N} \mathring{\vdash}^{-} \mathring{\Gamma} \qquad \mathring{M} \mathring{\vdash}^{-} \mathring{\Delta}}{\mathring{N} \oplus \mathring{M} \vdash^{-} \mathring{\Gamma} \check{\oplus} \mathring{\Delta}} \quad \frac{\mathring{X} \mathring{\vdash} \mathring{N} \check{\oplus} \mathring{M}}{\mathring{X} \mathring{\vdash} \mathring{N} \oplus \mathring{M}}\; \oplus_R$$

$$\oslash_L\; \frac{\mathring{P} \hat{\oslash} \mathring{N} \mathring{\vdash} \mathring{\Delta}}{\mathring{P} \oslash \mathring{N} \mathring{\vdash} \mathring{\Delta}} \quad \frac{\mathring{X} \mathring{\vdash}^{+} \mathring{P} \qquad \mathring{N} \mathring{\vdash}^{-} \mathring{\Delta}}{\mathring{X} \hat{\oslash} \mathring{\Delta} \vdash^{+} \mathring{P} \oslash \mathring{N}}\; \oslash_R \qquad \backslash_L\; \frac{\mathring{X} \mathring{\vdash}^{+} \mathring{P} \qquad \mathring{N} \mathring{\vdash}^{-} \mathring{\Delta}}{\mathring{P} \backslash \mathring{N} \vdash^{-} X \check{\backslash} \Delta} \quad \frac{\mathring{X} \mathring{\vdash} \mathring{P} \check{\backslash} \mathring{N}}{\mathring{X} \mathring{\vdash} \mathring{P} \backslash \mathring{N}}\; \backslash_R$$

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$$\downarrow_L\; \frac{N \vdash^{-} \Delta}{\downarrow N \cdot\!\vdash\!\cdot^{+} \check{\downarrow} \Delta} \quad \frac{\mathring{X} \mathring{\vdash}^{+} \check{\downarrow} N}{\mathring{X} \mathring{\vdash}^{+} \downarrow N}\; \downarrow_R \qquad \uparrow_L\; \frac{\hat{\uparrow} P \mathring{\vdash}^{-} \mathring{\Delta}}{\uparrow P \mathring{\vdash}^{-} \mathring{\Delta}} \quad \frac{X \vdash^{+} P}{\hat{\uparrow} X \cdot\!\vdash\!\cdot^{-} \uparrow P}\; \uparrow_R$$

$$\begin{array}{c} \hat{\otimes} \dashv \check{\backslash} \\ \\ \hat{\otimes} \dashv \check{/} \end{array}\; \begin{array}{c} \mathring{Y} \mathring{\vdash} \mathring{X} \check{\backslash} \mathring{\Delta} \\ \hline\hline \mathring{X} \hat{\otimes} \mathring{Y} \mathring{\vdash} \mathring{\Delta} \\ \hline\hline \mathring{X} \mathring{\vdash} \mathring{\Delta} \check{/} \mathring{Y} \end{array} \qquad \begin{array}{c} \mathring{X} \hat{\oslash} \mathring{\Delta} \mathring{\vdash} \mathring{\Gamma} \\ \hline\hline \mathring{X} \mathring{\vdash} \mathring{\Gamma} \check{\oplus} \mathring{\Delta} \\ \hline\hline \mathring{\Gamma} \hat{\obslash} \mathring{X} \mathring{\vdash} \mathring{\Delta} \end{array}\; \begin{array}{c} \hat{\oslash} \dashv \check{\oplus} \\ \\ \hat{\obslash} \dashv \check{\oplus} \end{array}$$

$$\begin{array}{c} \mathring{X} \mathring{\vdash} \Delta \\ \hline\hline \mathring{X} \mathring{\vdash}^{+} \check{\downarrow} \Delta \end{array}\; \check{\downarrow} \qquad \hat{\uparrow}\; \begin{array}{c} X \mathring{\vdash} \mathring{\Delta} \\ \hline\hline \hat{\uparrow} X \mathring{\vdash}^{-} \mathring{\Delta} \end{array}$$

A lot of rules!

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Visualizing the connections

See whether (and where) these rules **connect** to the rest of the calculus

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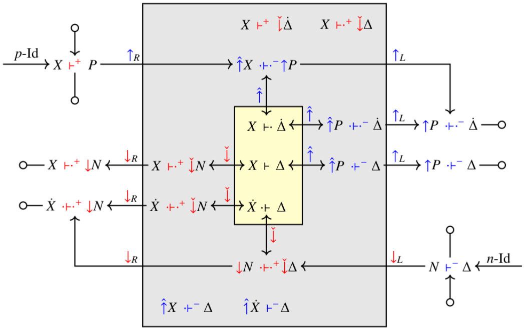


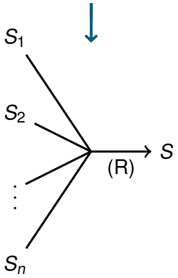
Figure 3 The topology of **fD.LG**-rules and phase transitions.

Identifying:

- **Zones** corresponding to phase
- **Crucial rules** mediating passing through these boundaries

Intuition about proof tree graphs

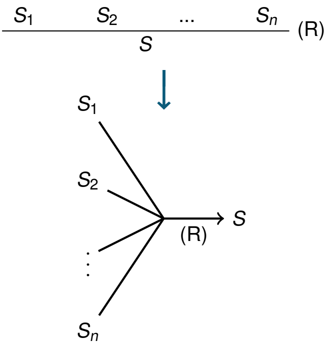
$$\frac{S_1 \qquad S_2 \qquad \dots \qquad S_n}{S}\ (\mathrm{R})$$



Proof tree graph (PTG):

- Vertices are **sets of sequents**
- Arcs are **rules**

Intuition about proof tree graphs



Proof tree graph (PTG):

- Vertices are **sets of sequents**
- Arcs are **rules**
- Dashed edges indicated **nonempty intersection**

$S - - - S'$ if $S \cap S' \neq \emptyset$

Goals:

- Broad **overview of the whole calculus**
- See which sequents are accessible

Let's build one together: Implicational logic

$$\frac{}{\varphi \vdash \varphi}\text{ Ax.} \qquad \frac{\Delta, \varphi \vdash \psi}{\Delta \vdash \varphi \to \psi} \to \text{I.} \qquad \frac{\Delta \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Delta, \Gamma \vdash \psi} \to \text{E.}$$

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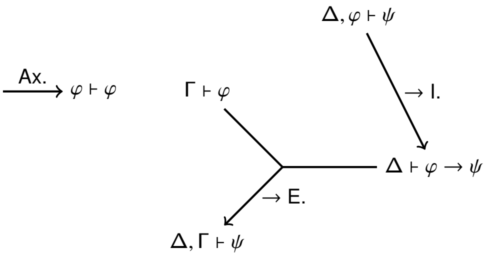

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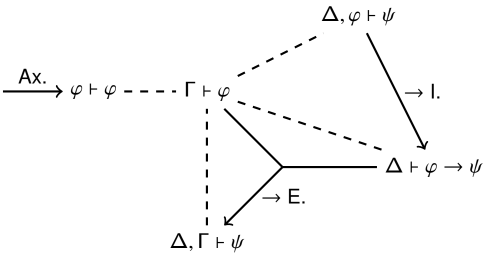

Formal definition

Set $\mathcal{K} = (\mathbb{S}, \mathcal{R})$ a **calculus**: signature (sorted function symbols) and rules

- $\mathcal{T}(\mathbb{S})$ well-formed sequents

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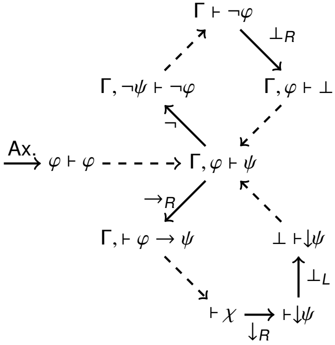
Definition

A **proof tree graph** on $\mathcal{K}$ is a hypergraph $G = (V, E, E_d)$

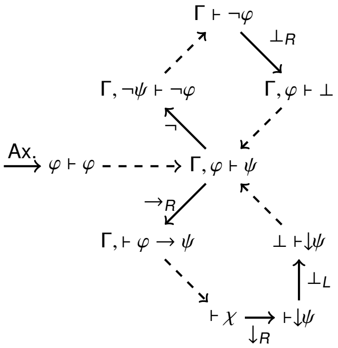
- $V \subseteq \wp(\mathcal{T}(\mathbb{S}))$
- $E \subseteq \bigcup_{n \geq 0} V^n \times \mathcal{R}_n \times V$
- $E_d \subseteq V \times V$

From graph to automaton



This PTG is just illustrative. The rules are invented.

From graph to automaton



- Like a non-deterministic **finite automaton** $\mathcal{A}$, e.g. Ax. $\rightarrow_R \downarrow_R \ \bot_L \neg \bot_R \neg \bot_R \in \mathcal{L}(\mathcal{A})$ ending on sequent

$$\neg \downarrow(\varphi \rightarrow \varphi), \neg\bot, \bot \vdash \bot$$

This PTG is just illustrative. The rules are invented.

Intuition about proof tree automata

Proof tree automaton (PTA) $\mathscr{A}$:

- non-deterministic finite tree automaton
- **States** are sets of sequents
- **Transitions** are rules
- ε**-transitions** are possible on nonempty intersections

Intuition about proof tree automata

Proof tree automaton (PTA) $\mathcal{A}$:

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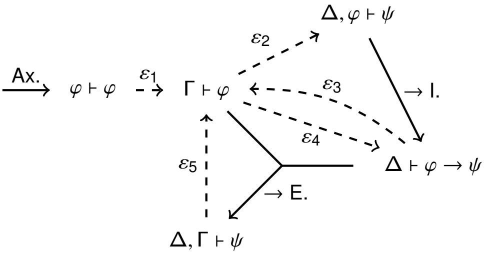
Goals:

- View backward proof search as **parsing**, i.e. finding a run on $\mathcal{A}$
- Establish a **correspondence** between operations on calculi and operations on tree automata

Tree automaton run



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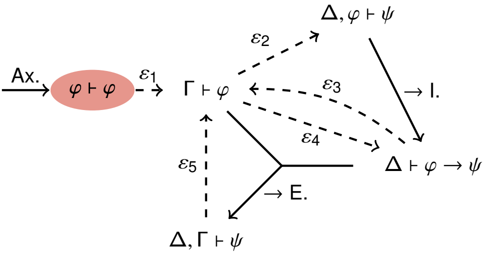
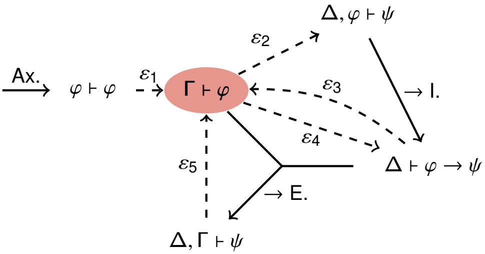


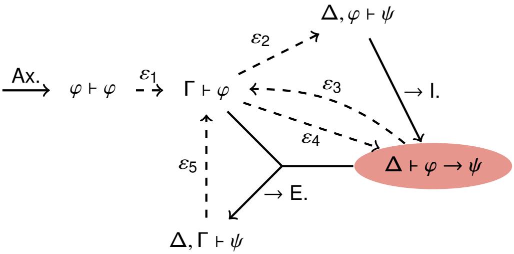
$$\frac{}{\boldsymbol{p \to q \vdash p \to q}}\ \text{Ax.}$$

Tree automaton run



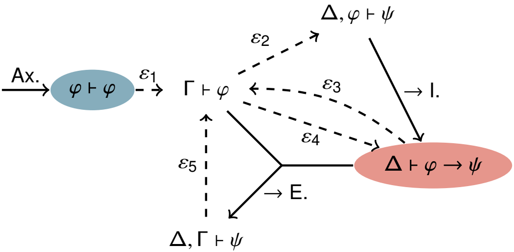
$$\dfrac{\dfrac{}{p \to q \vdash p \to q}\ \text{Ax.}}{p \to q \vdash p \to q}\ \varepsilon_1$$

Tree automaton run



$$\dfrac{\dfrac{\dfrac{}{p \rightarrow q \vdash p \rightarrow q}\ \text{Ax.}}{p \rightarrow q \vdash p \rightarrow q}\ \varepsilon_1}{p \rightarrow q \vdash p \rightarrow q}\ \varepsilon_4$$

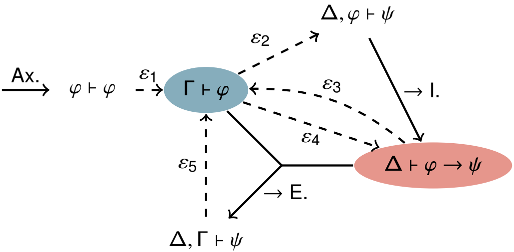
Tree automaton run



$$\frac{}{p \vdash p}\,\text{Ax.}$$

$$\cfrac{\cfrac{\cfrac{}{p \to q \vdash p \to q}\,\text{Ax.}}{p \to q \vdash p \to q}\,\varepsilon_1}{p \to q \vdash p \to q}\,\varepsilon_4$$

Tree automaton run



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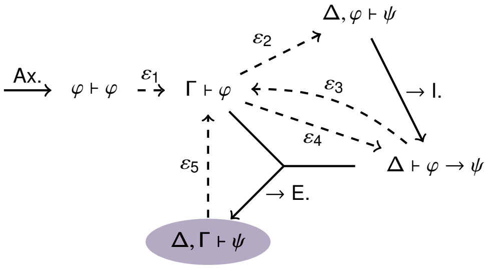
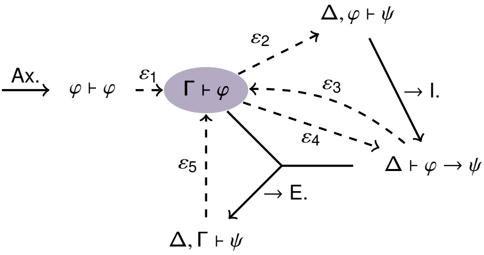


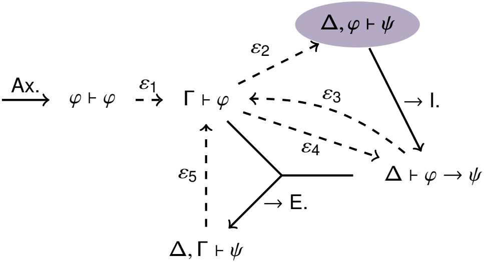
$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Ax.}}{p \vdash p}\,\varepsilon_1 \qquad \dfrac{\dfrac{\dfrac{}{p \to q \vdash p \to q}\,\text{Ax.}}{p \to q \vdash p \to q}\,\varepsilon_1}{p \to q \vdash p \to q}\,\varepsilon_4}{p, p \to q \vdash q}\,\to\text{E.}$$

Tree automaton run



$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Ax.}}{p \vdash p}\,\varepsilon_1 \qquad \dfrac{\dfrac{\dfrac{}{p \to q \vdash p \to q}\,\text{Ax.}}{p \to q \vdash p \to q}\,\varepsilon_1}{p \to q \vdash p \to q}\,\varepsilon_4}{\dfrac{p, p \to q \vdash q}{p, p \to q \vdash q}\,\varepsilon_5}\,\to \text{E.}$$

Tree automaton run



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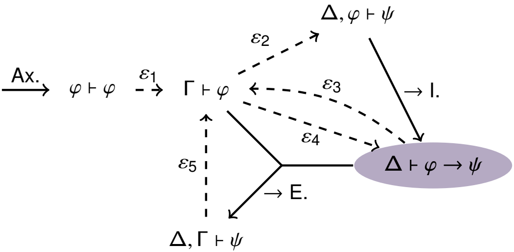
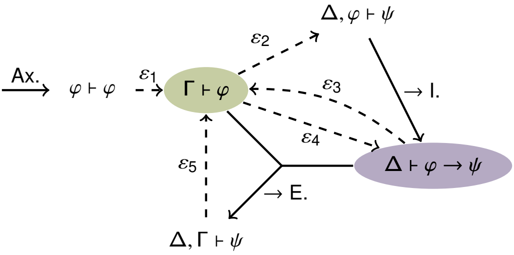
\dfrac{\dfrac{}{\underline{p} \vdash \underline{p}}\ \text{Ax.}}{p \vdash p}\ \varepsilon_1
\qquad
\dfrac{\dfrac{\dfrac{}{\underline{p} \rightarrow \underline{q} \vdash \underline{p} \rightarrow \underline{q}}\ \text{Ax.}}{\underline{p} \rightarrow \underline{q} \vdash \underline{p} \rightarrow \underline{q}}\ \varepsilon_1}{p \rightarrow q \vdash p \rightarrow q}\ \varepsilon_4
}{
\dfrac{\dfrac{\underline{p}, \underline{p} \rightarrow \underline{q} \vdash \underline{q}}{\underline{p}, \underline{p} \rightarrow \underline{q} \vdash \underline{q}}\ \varepsilon_5}{p, p \rightarrow q \vdash q}\ \varepsilon_2
}\ \rightarrow \text{E.}
$$

Tree automaton run



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Ax.}}{p \vdash p}\,\varepsilon_1 \qquad \dfrac{\dfrac{\dfrac{}{p \to q \vdash p \to q}\,\text{Ax.}}{p \to q \vdash p \to q}\,\varepsilon_1}{p \to q \vdash p \to q}\,\varepsilon_4}{p, p \to q \vdash q}\,\to \text{E.}}{p, p \to q \vdash q}\,\varepsilon_5}{p, p \to q \vdash q}\,\varepsilon_2}{p \to q \vdash p \to q}\,\to \text{I.}$$

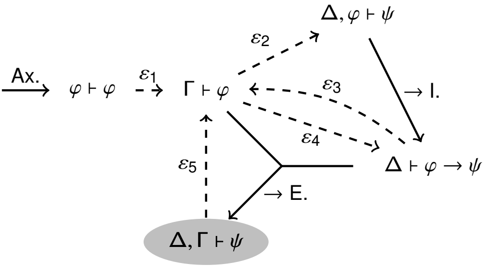
Tree automaton run



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Ax.}}{p \vdash p}\,\varepsilon_1 \qquad \dfrac{\dfrac{\dfrac{}{p \to q \vdash p \to q}\,\text{Ax.}}{p \to q \vdash p \to q}\,\varepsilon_1}{p \to q \vdash p \to q}\,\varepsilon_4}{p, p \to q \vdash q}\,\to \text{E.}}{p, p \to q \vdash q}\,\varepsilon_5}{p, p \to q \vdash q}\,\varepsilon_2}{p \to q \vdash p \to q}\,\to \text{I.}$$

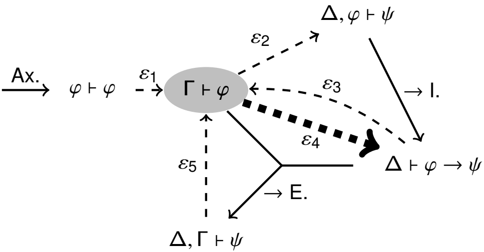
$$\dfrac{\dfrac{}{(p \to q) \to q \vdash (p \to q) \to q}\,\text{Ax.}}{(p \to q) \to q \vdash (p \to q) \to q}\,\varepsilon_1$$

Tree automaton run



$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{p \vdash p}\,\text{Ax.}}{p \vdash p}\,\varepsilon_1
\qquad
\dfrac{\dfrac{\dfrac{}{p \rightarrow q \vdash p \rightarrow q}\,\text{Ax.}}{p \rightarrow q \vdash p \rightarrow q}\,\varepsilon_1}{p \rightarrow q \vdash p \rightarrow q}\,\varepsilon_4
}{p, p \rightarrow q \vdash q}\,\rightarrow\text{E.}
}{p, p \rightarrow q \vdash q}\,\varepsilon_5
}{p, p \rightarrow q \vdash q}\,\varepsilon_2
}{p \rightarrow q \vdash p \rightarrow q}\,\rightarrow\text{I.}
\qquad
\dfrac{\dfrac{}{(p \rightarrow q) \rightarrow q \vdash (p \rightarrow q) \rightarrow q}\,\text{Ax.}}{(p \rightarrow q) \rightarrow q \vdash (p \rightarrow q) \rightarrow q}\,\varepsilon_1
}{p \rightarrow q, (p \rightarrow q) \rightarrow q \vdash q}\,\rightarrow\text{E.}
$$

Tree automaton run



$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{p \vdash p}\;\text{Ax.}}{p \vdash p}\;\varepsilon_1
      \qquad
      \dfrac{\dfrac{\dfrac{}{p \to q \vdash p \to q}\;\text{Ax.}}{p \to q \vdash p \to q}\;\varepsilon_1}{p \to q \vdash p \to q}\;\varepsilon_4
    }{p, p \to q \vdash q}\;\to\text{E.}
  }{\dfrac{p, p \to q \vdash q}{p, p \to q \vdash q}\;\varepsilon_2}\;\varepsilon_5
  }{p \to q \vdash p \to q}\;\to\text{I.}
  \qquad
  \dfrac{\dfrac{}{(p \to q) \to q \vdash (p \to q) \to q}\;\text{Ax.}}{(p \to q) \to q \vdash (p \to q) \to q}\;\varepsilon_1
}{
  \dfrac{\dfrac{p \to q, (p \to q) \to q \vdash q}{p \to q, (p \to q) \to q \vdash q}\;\varepsilon_5}{}\;\not\varepsilon_4
}\;\to\text{E.}
$$

The crucial role of control

ε-transitions are not always allowed:

- Depending on the instance sequent
- **Instance sequents** are changes by rules

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Proof tree automaton $\mathcal{A}$ = regular tree automaton $F(\mathcal{A})$

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Proof tree automaton $\mathcal{A}$ = regular tree automaton $F(\mathcal{A})$

$\oplus$ **control relations** ∇ and ∇_ε on instances

- Run in $F(\mathcal{A})$ = free walk in the graph with no restriction
- $\mathcal{L}(\mathcal{A}) \subseteq L(F(\mathcal{A}))$
- $D \in \mathcal{L}(F(\mathcal{A}))$ is correct if $D \in \mathcal{L}(\mathcal{A})$

Decomposition

Decomposing $\mathcal{A}$ as a **functor** $U : \mathcal{K} \to F(\mathcal{A})$

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Proposition

$D \in F(\mathcal{A})$ is correct iff D belong to the image of U

U is a monoidal refinement system (Melliès and Zeilberger 2015)

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See full definitions, proofs an other relevant properties on arXiv...

Conclusion

Proof tree graph = Novel tool to visualize whole (or a part of a) calculus

→ *try it yourself!*

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Eventually: **Correspondence** between properties on

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- Proof tree languages
- Graphs (e.g. topological arguments, rewriting techniques,...)
- Tree automata

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$\longrightarrow$ *try it yourself!*

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- Proof tree languages
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Future plan:

- μ-calculus to express control properties using tree structure only

Thank you!

Bretto, Alain (17th Apr. 2013). **Hypergraph Theory: An Introduction**. Springer Science & Business Media. 129 pp. ISBN: 978-3-319-00080-0. Google Books: `lb5DAAAAQBAJ`.

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Melliès, Paul-André and Noam Zeilberger (15th Jan. 2015). "Functors Are Type Refinement Systems". In: 42nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2015). DOI: `10.1145/2676726.2676970`. URL: `https://hal.inria.fr/hal-01096910` (visited on 24/03/2021).

More examples

$$
\begin{array}{rrl}
A \ni & u, v ::= & r(s) \mid f(s) \mid a \\
B \ni & s, t ::= & l(u) \mid g(u) \\
C \ni & h ::= & u \;{}_{A}{\vdash}_{A}\; v \mid u \;{}_{A}{\vdash}_{B}\; t \mid s \;{}_{B}{\vdash}_{B}\; t
\end{array}
$$

$$
\frac{}{u \;{}_{A}{\vdash}_{A}\; u}\ (\mathrm{Ax}) \qquad \frac{u \;{}_{A}{\vdash}_{A}\; r(t)}{l(u) \;{}_{B}{\vdash}_{B}\; t}\ (\mathrm{Ad})
$$

$$
\frac{u \;{}_{A}{\vdash}_{B}\; g(v)}{u \;{}_{A}{\vdash}_{A}\; v}\ (\mathrm{g'}) \qquad \frac{s \;{}_{B}{\vdash}_{B}\; t}{f(s) \;{}_{A}{\vdash}_{B}\; t}\ (f)
$$